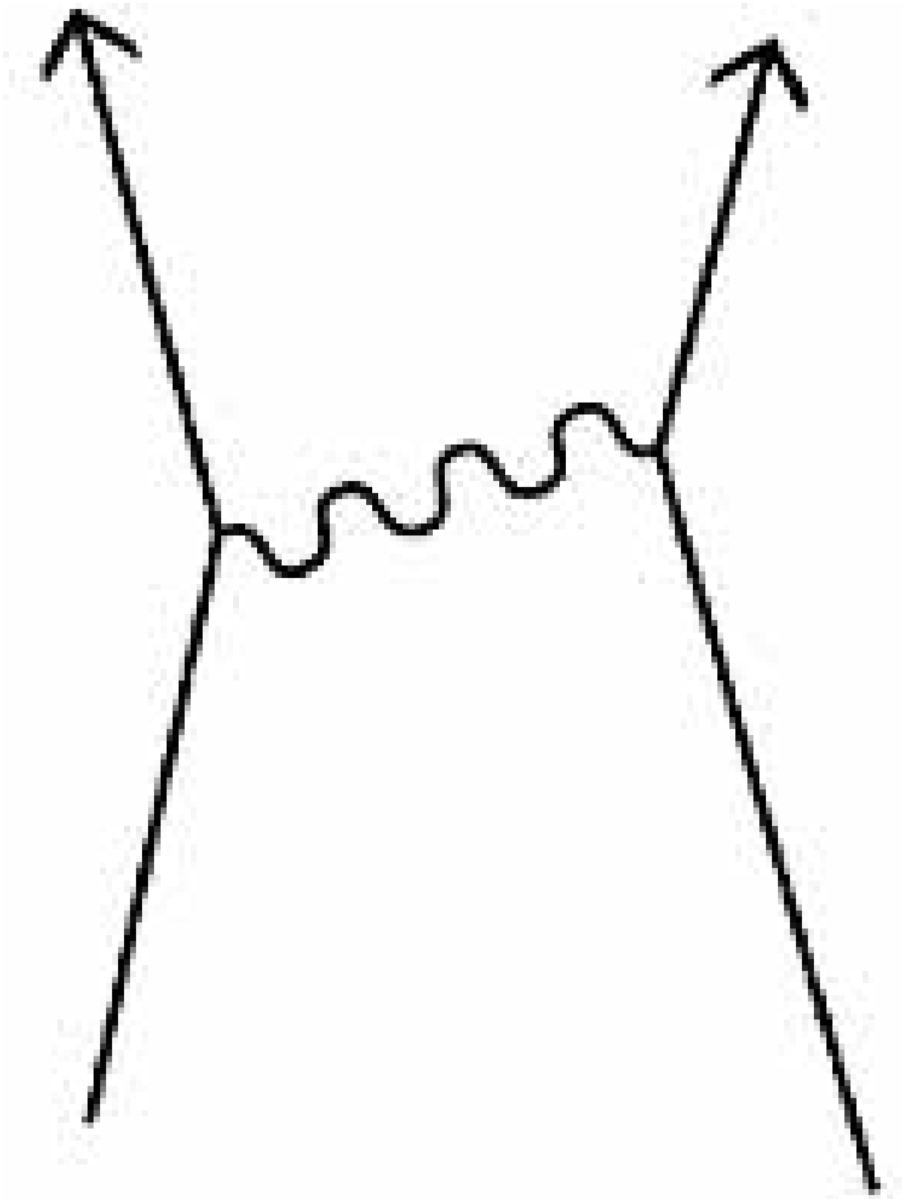
The Early Universe

Peter L.

10th March 2006



From the weak interaction cross section

$$\sigma_{\mathrm{weak}} = \frac{\alpha^2 T^2}{m_W^4}, \tag{1}$$

where $\alpha = 1/128$, T is the temperature and $M_W \simeq 80$ GeV is the mass of the W boson, we get the interaction rate

$$\begin{aligned} \Gamma &\equiv n\,\sigma_{\mathrm{weak}}|v| \\ &\sim= \frac{\alpha^2}{m_W^4} T^5, \end{aligned} \tag{2}$$

since $n \sim 1/a^3 \sim T^3$, and $|v| \sim c \equiv 1$.
Furthermore, since the Hubble time is

$$\begin{aligned} H &= 1.66 g_\star^{1/2} \frac{T^2}{m_{\mathrm{Pl}}} \\ &\sim \frac{T^2}{m_{\mathrm{Pl}}}, \end{aligned} \tag{3}$$

where $m_{\mathrm{Pl}} \sim 10^{19}$ GeV is the Planck mass, we get that

$$\frac{\Gamma}{H} \sim \frac{\alpha^2}{m_W^4} m_{\mathrm{Pl}} T^3, \tag{4}$$

so that, at the time of freeze out, where $\Gamma/H \sim 1$,

$$\begin{aligned} T &\sim \left(\frac{M_W^4}{\alpha^2 m_{\mathrm{Pl}}} \right)^{1/3} \\ &\sim 4 \text{ MeV}. \end{aligned} \tag{5}$$

Figure 1 shows the relation of Γ/H as a function of $1/T$.
Dividing the equation for the dark matter density parameter Ω_{DM},

$$\Omega_{\mathrm{DM}} h^2 = 7.8 \times 10^{-2} \frac{g_{\mathrm{eff}}}{g_\star^s(x_f)} \frac{m}{\mathrm{eV}}, \tag{6}$$

where $h \simeq 0.7$ for a ΛCDM cosmology $g_{\mathrm{eff}} = 3g/4 = 3/2$ for fermions, $g_\star^s(x_f) = 10.75$, and m is the particle mass, in this case the neutrino, with the equation for the present abundance Y_∞ of the neutrinos,

$$\begin{aligned} Y_\infty &= 0.278 \frac{g_{\mathrm{eff}}}{g_\star^s(x_f)} \\ &= 0.039, \end{aligned} \tag{7}$$

we get

$$\frac{\Omega_{\mathrm{DM}} h^2}{Y_\infty} = 0.28 \frac{m}{\mathrm{eV}} \tag{8}$$

$$\frac{m}{\mathrm{keV}} \sim 10. \tag{9}$$

However, since it is believed that dark matter particles must obey $m \gtrsim 0.5$ keV, the neutrino is not a very good candidate.

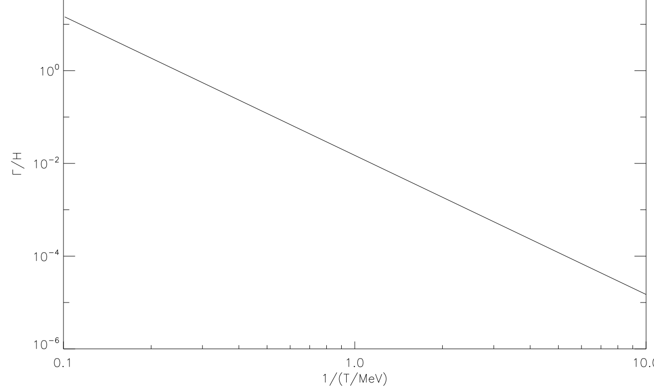


Figure 1: The evolution of Γ/H with $1/T$, which essentially corresponds to time.

2 Sterile neutrinos – non-thermal production

Active neutrinos $\nu_{\rm a}$ and sterile neutrinos $\nu_{\rm s}$ mix through their mixing angle θ in vacuum according to

$$\begin{pmatrix} \nu_{\rm a} \\ \nu_{\rm s} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}, \tag{10}$$

where ν_1 and ν_2 are the light and heavy mass eigenstates, respectively.
In matter, the mixing angle $\theta_{\rm M}$ is given by

$$\sin 2\theta_{\rm M} = \frac{\sin 2\theta}{1 + 2.27\times 10^{-20}(m/\mathrm{MeV})^{-2}y^2/x^6}, \tag{11}$$

where m is the heavy mass (the light mass is approximately zero), $x = (1\ \mathrm{Mev})a$, $y^2 = (Ea)^2 \sim 10$, and $a \sim 1/T$ is the scale factor. Thus, for matter effects to become important,

$$\begin{aligned} 1 &\sim 2.27\times 10^{-20}\left(\frac{1}{m/\mathrm{MeV}}\right)^2 \frac{10}{(\mathrm{MeV})^6 a^6} \\ &= 2.27\times 10^{5}\left(\frac{1}{m/\mathrm{keV}}\right)^2 \left(\frac{T}{\mathrm{GeV}}\right)^6, \end{aligned} \tag{12}$$

or

$$T = 0.13\ \mathrm{GeV}\left(\frac{m}{\mathrm{keV}}\right)^{1/3}. \tag{13}$$

In matter, the production of the sterile neutrinos are reduced by a factor of $(\sin^2 2\theta_{\rm M})/4$. Figure 2 again shows the relation of Γ/H as a function of $1/T$, but now including the reducing. We note that Γ/H now reaches a maximum of $\sim 10^{-4}$, implying that the sterile neutrinos do not reach equilibrium.
Figure 3 shows the result of varying the mixing angle and the mass. We see that Γ/H increases as mixing angle and mass increases. Still, however, it would require an unrealistically large value of either, or both, in order for Γ/H to reach unity.
The evolution of the Boltzmann equation is governed by the relation

$$\widehat{\mathbf{L}} f_{\rm s} = \widehat{\mathbf{C}} f_{\rm s}, \tag{14}$$

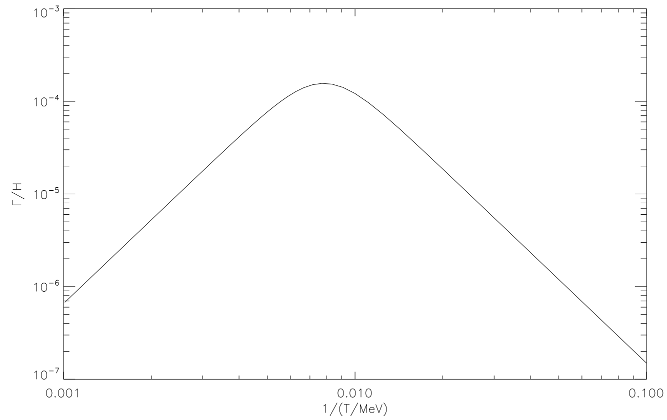


Figure 2: The evolution of Γ/H with $1/T$, taking into account the effect of matter reducing the production rate. Now Γ/H does not reach ~ 1, so there is no freeze out.

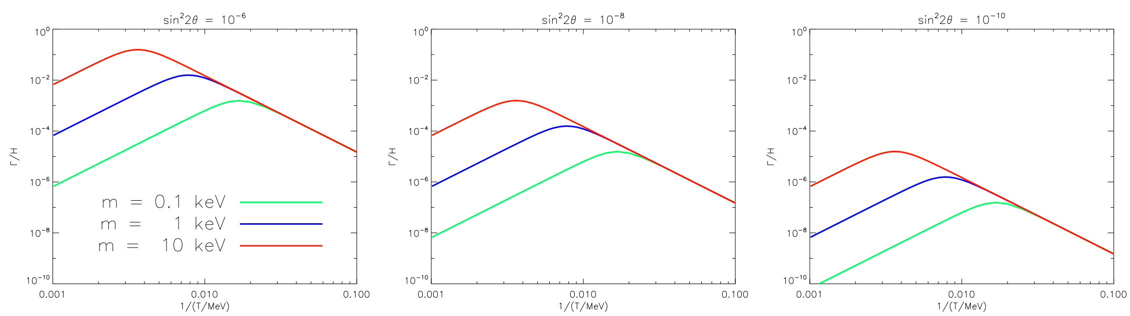


Figure 3: The evolution of Γ/H with $1/T$, for different values of mixing angle θ and mass m. It seems impossible to reach $\Gamma/H \sim 1$ with reasonable values of θ and m.

function. Since collisions are in essence given by the interaction rate, we can write

$$\begin{aligned}\widehat{\mathbf{C}}f_{\rm s} &= \Gamma_{\rm s}f_{\rm s} \\ &= \frac{\sin^2 2\theta_{\rm M}}{4}\Gamma_{\rm a}f_{\rm a}.\end{aligned} \tag{15}$$

The Liouville operator is given by

$$\widehat{\mathbf{L}} = \partial_t - pH\partial_p, \tag{16}$$

where ∂_i is the derivative with respect to variable i, and p is momentum.
From the chain rule, the time derivative of $f_{\rm s}$ is

$$\partial_t f_{\rm s} = \dot{x}\partial_x f_{\rm s} + \dot{y}\partial_y f_{\rm s}, \tag{17}$$

and, since $y = Ea \simeq pa$ for ultra-relativistic particles, the momentum derivative of $f_{\rm s}$ is

$$\partial_p f_{\rm s} = a\partial_y f_{\rm s}, \tag{18}$$

so that

$$\widehat{\mathbf{L}} = \dot{x}\partial_x f_{\rm s} + (\dot{y} - pHa)\partial_y f_{\rm s}. \tag{19}$$

However, due to p being a constant of the motion,

$$\begin{aligned}\dot{y} &= \dot{p}\partial_p y + \dot{a}\partial_a y \\ &= \dot{a}p.\end{aligned} \tag{20}$$

Thus, since $H = \dot{a}/a = \dot{x}/x$, the terms in the brackets in Eq. 19 cancel, and, combining Eqs. 15 and 19,

$$xH\partial_x f_{\rm s} = \frac{\sin^2 2\theta_{\rm M}}{4}\Gamma_{\rm a}f_{\rm a}. \tag{21}$$

$f_{\rm s}$ gives us the distribution of sterile neutrinos in phase space. To get the density, we integrate over all momenta:

$$n_{\rm s} = \int_0^\infty f_{\rm s} d^3p. \tag{22}$$

Expressing $f_{\rm s}$ as a function of x and y essentially hides the expansion of space, so that $f_{\rm a}$ exits the integration, and, integrating Eq. 21 with the expression for $\Gamma_{\rm a}/H$ from Eq. 4 and $\sin^2 2\theta_{\rm M}$ from Eq. 11,

$$f_{\rm s} = c_1 f_{\rm a}\int_0^\infty \frac{1}{x^4}\left(\frac{1}{1 + c_2/x^6}\right)^2 dx, \tag{23}$$

where

$$c_1 = \frac{\sin^2 2\theta}{4}\frac{\alpha^2}{m_W^4}m_{\rm Pl}, \tag{24}$$

and

$$c_2 = 2.27 \times 10^{-19}\left(\frac{m}{\rm MeV}\right)^{-2}. \tag{25}$$

$$
\begin{aligned}
f_{\rm s} &= \frac{c_1}{3} f_{\rm a} \int_0^\infty \left(\frac{1}{1 + c_2/\chi^2} \right)^2 d\chi & (26) \\
&= \frac{4}{3\pi} f_{\rm a} \frac{c_1}{\sqrt{c_2}}. & (27)
\end{aligned}
$$

Thus, Eq. 22 becomes

$$
n_{\rm s} = \frac{4}{3\pi} n_{\rm a} \frac{c_1}{\sqrt{c_2}}. \quad (28)
$$

With $m = 1$ keV and $\sin^2 2\theta = 10^{-8}$ this equates to $n_{\rm s} = 4 \times 10^{-3}$, or, with $\rho_{\rm cr} \sim 10h^2$ keV cm^{-3},

$$
\begin{aligned}
\Omega_{\rm s} h^2 &= \frac{n_{\rm s} m}{\rho_{\rm cr}} \\
\Omega_{\rm s} &= 1.5 \times 10^{-3}. & (29)
\end{aligned}
$$

We can now express $\sin^2 2\theta$ as a function of m with $\Omega_{\rm s}$ a varying parameter:

$$
\begin{aligned}
\sin^2 2\theta &= 3\pi \frac{\rho_{\rm cr} h^2}{n_{\rm a}} \frac{\sqrt{2.27 \times 10^{-19}}}{\alpha^2 m_{\rm Pl}/m_W^4} \Omega_{\rm s} m^{-2} & (30) \\
&= 6.6 \times 10^{-12} \Omega_{\rm s} m^{-2}, & (31)
\end{aligned}
$$

and, equating $\Omega_{\rm s}$ to $\Omega_{\rm DM} = 0.29 \pm 0.07$, make a nice contour plot. Figure 4 shows the relation between $\sin^2 2\theta$ and m with constraints from $\Omega_{\rm DM}$. Also, constraints from observational exclusion lines ($m > 0.5$ keV) and from the diffuse gamma background ($\sin^2 2\theta < 2.5 \times 10^{-19} (m/\text{MeV})^{-5}$) are shown; we see that the sterile neutrino is actually a candidate for the dark matter.

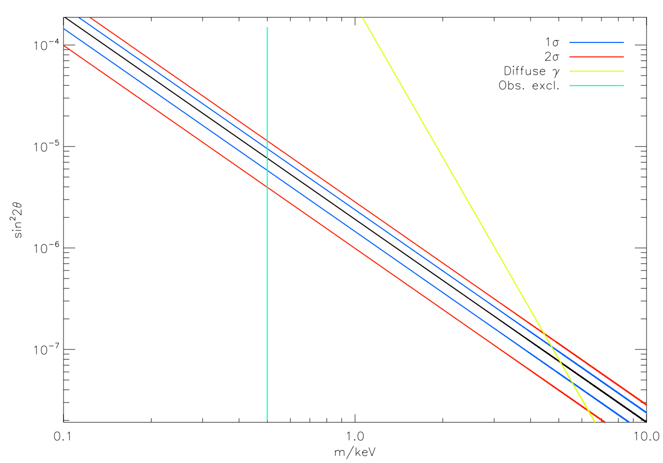


Figure 4: 2σ contour plot in $(m, \sin^2 2\theta)$ parameter space. The black line shows $\sin^2 2\theta$ as a function of m, with $1\sigma_\Omega$ (blue) and $2\sigma_\Omega$ (red). Constraints from observational exclusion lines (cyan) and diffuse gamma background (bile) are also displayed. Clearly, there is still space left for the sterile neutrino as dark matter.